Deepening the content knowledge of area and perimeter with GeoGebra

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Abstract

The article focuses on deepening the content knowledge of area and perimeter at secondary school via an isoperimetric issue supported by GeoGebra diagrams. The text introduces the isoperimetric problem, introduces and analyzes the survey held on the isoperimetric issue at Czech secondary schools, and provides an ICT support to the secondary school instruction related to the isoperimetric problem: GeoGebra diagrams on area and perimeter interdependence, with a track to volume and surface upgrade.

1 Introduction

At last they landed, where from far your eyes
May view the turrets of new Carthage rise;
There bought a space of ground, which (Byrsa call'd,
From the bull's hide) they first inclos'd, and wall'd.
Virgil, The Aeneid

According to an ancient Greek legend recorded by Virgil, the Phoenician queen Dido fled from Tyre to North Africa and bought there a land from local inhabitants. The contract was for a land which can be bounded by the hide of a bull. The locals thought the contract as a joke, but Dido cut the hide into a lot of narrow strips, and made a long rope from them. Then she laid the rope to form a semi-circle along the seashore, and enclosed the whole Byrsa hill by the bulls hide. That is the way how Dido got the land for founding the city of Carthage.

Regardless of whether this Virgils tale is true or not, it is important for us that more than 2000 years ago at least Virgil knew that the semicircle is the best choice for obtaining the largest possible land.

What about our students? Do they have the same knowledge? What shape would they enclose by the rope? Do they know which shape is the best for obtaining maximum area?

2 The survey

During the survey we queried 383 lower- and upper-secondary school students of various types of Czech schools (a basic school, a vocational school, and a grammar school), and of various ages (from 12 to 19 yrs). Students were not individually selected; the survey was conducted in whole classes.

Since Czech students are not used to manipulate with ox-hides, the original historical task would be a little confusing for them. This was the reason why we decided to focus just on the second part of the Dido's task, and updated it to meet more of our everyday reality. The final formulation of the survey question arose as:

"You have 40 meters of wire fence, which shape do you choose to enclose the largest land?"

The main intention was to check students' basic knowledge, so that we decided for a flash survey. The survey was a matter of one minute: students were given small pieces of paper, wrote a one-word answer immediately, and return the paper back to the teacher. Then the students informally discussed the issue with the teacher.

We got the following percentage of answers:

40%	for circle,
27%	for square,
20%	for rectangle,
10%	for "doesn't matter,"
1%	for n-gon,
2%	altogether for various other shapes (triangle, rhombus, ellipse, oval, trapezoid).

No semicircles. No Dido among the students.

3 Mathematical background of the survey

Finding a bounded land with the biggest possible area is a part of the so-called isoperimetric problem, which generally seeks a curve of fixed length that connects two points on another given curve in such a way that the newly formed closed curve bounds the largest area.

In Dido's case, the given curve is the line of the seashore, and the fixed length is the length of the rope. The seashore can be understood as an external border: the rope was made from the bulls hide, which was a part of the contract, but the seashore was not mentioned in the contract at all. Dido had to do a decision to use an external border, and choose which one.

The situation without external borders corresponds to a case when the given curve is a single point – in this case the isoperimetric problem seeks a closed curve of fixed length which bounds the largest area. Ancient Greek mathematicians knew this problem, they solved it partially: Zenodorus proved that a circle has greater area than any polygon with the same perimeter.

The complete solution to the isoperimetric problem is not easy, even for cases when the given curve is a line or a single point (for details see [1], [5], and sources mentioned there). There are many different ways how to approach the solution — by algebraic calculations, by geometrical manipulations, by means of calculus of variations, by vector analysis connected to physics, by integral geometry, by Fourier series manipulation, by means of complex plane calculus, by means of functional analysis, etc.

We shall briefly introduce a solution based on the work of Steiner. Steiner himself proved the following theorem:

Theorem 1 *Any figure with maximal area must be a circle.*

The proof of the theorem was published in [7], its English version is taken from [1]. Detailed commentary to the proof can be found in [5].

Proof. Take a figure with maximal area, and cut its perimeter in half with a line (Fig. 1a). This line will split the area in half as well, because if it did not we could take the half with the greater area together with its reflection in the line and get a figure with the same perimeter but greater area.

Consider one of these halves. Suppose it is not a semicircle. Then there will be some point on the boundary where lines drawn from the points on the symmetry line meet at an angle that is not a right angle (Fig. 1b).

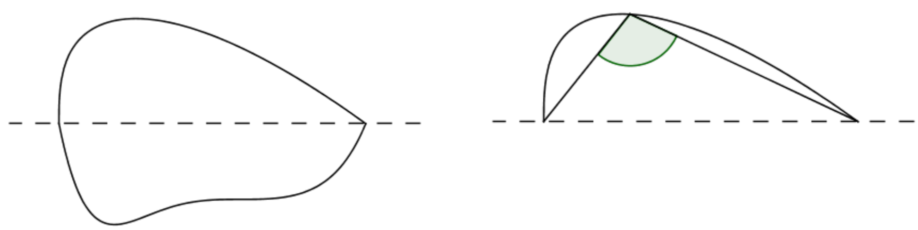

Figure 1: A figure with perimeter cut in half (a), an angle that is not right (b).

Think of there being a void inside the triangle and think of the pieces on the sides as glued on. Slide the endpoints along the symmetry line to make the angle a right angle.

This increase the area, so reflecting this gives a figure with greater area while the perimeter is still the same. That is impossible, so the halves must be semicircles and our figure must have been a circle to begin with. ■

As you can see, Steiner's approach does not solve the problem completely, it lacks verification of existence of a figure with maximal area. This was covered by Carathéodory and Study sixty years later:

Theorem 2 *The method of Steiner's proof converges to the circle.*

The proof of the theorem was published in [2], its English version is taken from [1] and shortened.

Proof. As in Steiner's proof, we work with half-figures, curves of length L that start and end on a given line. For such a curve, take the convex hull — this increases the area and decreases the perimeter. Since we need the perimeter unchanged, we rescale (= enlarge) the convex hull so that it has length L. Then apply Steiner's procedure from Theorem 1. We repeat these two steps and always keep the left endpoint fixed. We claim that this process converges uniformly to a semicircle of length L.

When we change the angle indicated in Fig. 1b to a right angle, then the area is increased by

$$\frac{|\text{first side}| \cdot |\text{second side}|}{2} \cdot \big(1 - \sin(\text{ the angle})\big)$$

This quantity varies continuously along the curve, so it attains its maximum, and we can agree always to take the first point where the maximum is attained as the point at which we apply our Steiner's improvement.

Now consider the sequence of right endpoints. These stay bounded, so they have at least one limit point. Extract a subsequence of curves for which the right endpoints converge. Consider an epsilon-thin strip around the semicircle that begins at the left point and ends at this right limit point. Figures constructed by this method will be forced, after a finite number of steps, to stay completely inside the strip. The semicircle that we are approximating uniformly with curves of length L must be the semicircle of length L. ■

Summarized, in the line case the right answer to the isoperimetric problem consists of a semicircle (this affirms that Dido was right), and in the point case the right answer is a circle.

After several simply calculations we find out the full range of Dido's ingenuity: with the same length of the rope, the area of the semicircle equals twice the area of the circle. Subsequently, the area of a quadrant would equal fourfold the area of the circle. See Fig. 2.

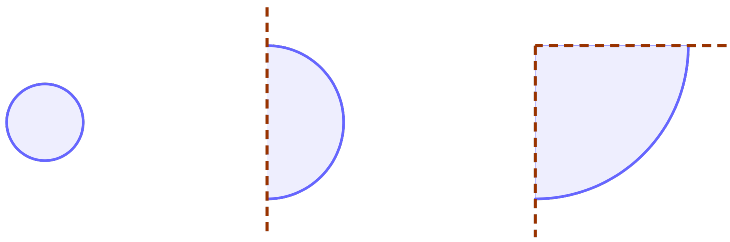

Figure 2: A circle, a semicircle, and a quadrant created with the same length of the rope.

As shown above, the complete solution to the isoperimetric problem is outside the secondary school curriculum; we cannot present the proof to our pupils. But we can easily give them an insight into the problem, show them some aspects. For example, it is easy to prove that for a given square, the circle with the same perimeter has bigger area – see Fig. 3.

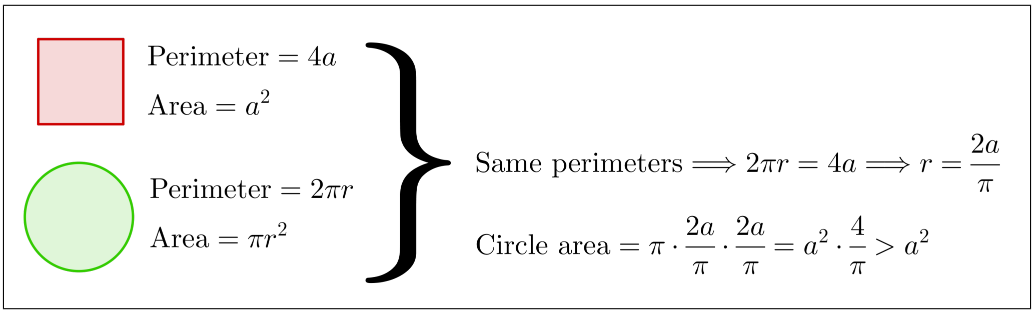

Figure 3: For a given square the circle with the same perimeter has bigger area.

Similarly we can replace the square by an arbitrary plane figure with simple and easily comparable formulas on perimeter and area (an equilateral triangle, a right-angled isosceles triangle, a regular hexagon, a regular octagon, a regular dodecagon, etc.), and find out that the area of a circle is bigger than the area of this figure. The comparison of a circle and a rectangle is more difficult, it requires the AM-GM inequality.

4 Analyzing the survey

Let's focus on the answers obtained from the survey. The most frequent answer was the circle answer. This answer is correct. It is also a correct answer to a related geometric question "Which plane figure has the biggest area for a given perimeter?"

But the circle answer has only 40% percentage. Does it mean that the remaining 60% of answers were wrong?

During post-survey discussions some students (successfully) defended their non-circle answers with interesting arguments which together initiate the following summary:

- The circle answer is not correct. Just try to imagine several nearest neighbors who all want to have a circle land as in Fig. 4. Who would take care of the remaining land pieces outside the circles?

- It is not possible to build a perfectly circular wire fence, because the wire needs to be attached to posts. We have to find a reasonable substitute such as a regular polygon with an appropriate number of sides. For our 40 meters' fence, 40-gon shape would be fine. Somebody may prefer a regular hexagon, for better tessellation.

- If you observe a cadastral map, you can see that most of the lands are quadrangular. Among all quadrangulars, the square is the one with the largest area for a given perimeter. With a straight neighbors fence forming one border of the land, the rectangle shape is the best choice.

These students made up about 20% of the total. The remaining 40% of students just did not know a correct answer.

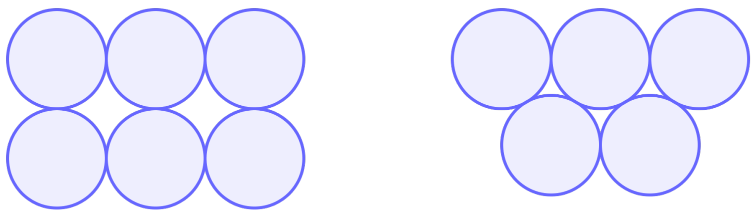

Figure 4: Neighboring circle lands.

Can we help teachers to lower that 40% of incorrect answers?

5 ICT support to related instruction

The knowledge of the isoperimetric problem and of other relations between area and perimeter can be successfully supported by suitable software. We shall demonstrate some of this support with GeoGebra, free mathematics software for teaching and learning (available at [3]). GeoGebra is an intuitive tool with easy handling, not difficult for pupils and teachers of any age. It is important for us that GeoGebra commands can return perimeter and area of polygons, circles, and ellipses. For our comfort, GeoGebra also allows writing interactive texts with perimeter or area values.

The illustration to the isoperimetric problem can start with a square and a circle placed to a common picture as in Fig. 5. This picture illustrates two facts: for a given perimeter circles have bigger area than squares, and for a given area squares have bigger perimeter than circles. In GeoGebra environment we can change values of area/perimeter arbitrarily by a slider to show that these rules apply to circles and squares of any size.

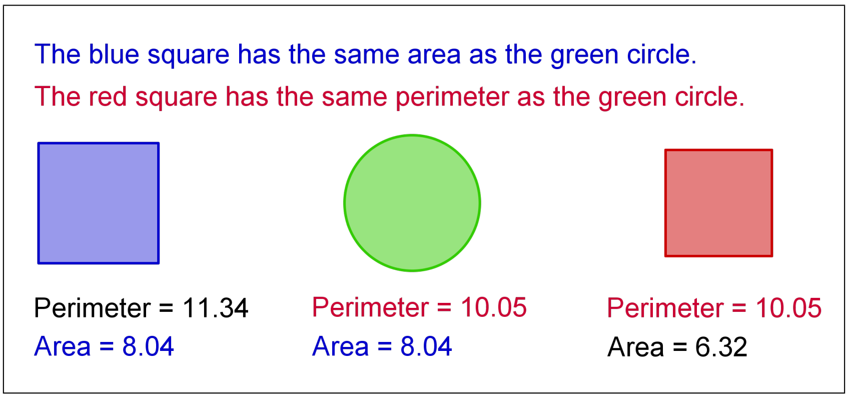

Figure 5: Square and circle together, with area and perimeter values.

Similar pictures can be created for various pairs of plane shapes.

Another option is to mix together several different shapes with the same area or perimeter as in Fig. 6 and 7.

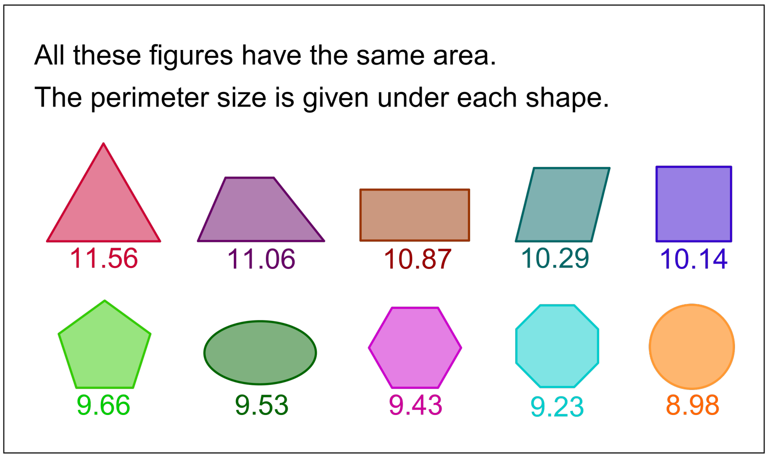

Figure 6: Various plane figures with the same area.

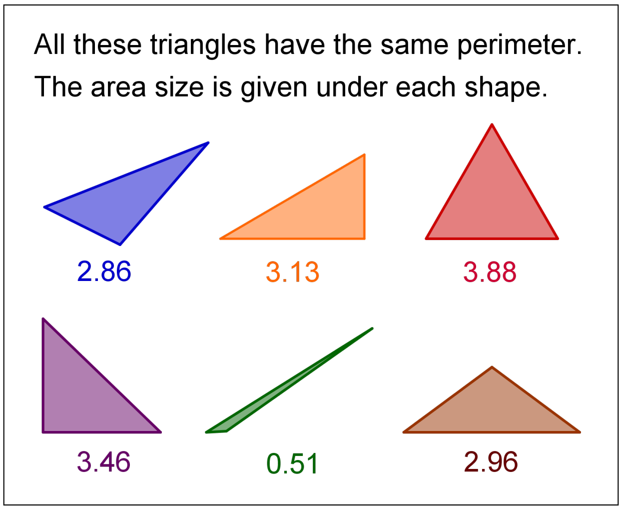

Figure 7: Various triangles with the same perimeter.

The GeoGebra diagrams presented here are not complicated; they can be created even by students themselves. These diagrams use only basic formulas on area and perimeter (circumference) of plane figures, basic rules for their construction, and GeoGebra commands Polygon, Circle, Ellipse, Area, and Circumference.

6 Related problems in 3D

Since we have skilled GeoGebra developers who work on the 3D upgrade — beta version of GeoGebra 5.0 is already available at [4] — we can move from plane to space, and create similar GeoGebra diagrams on volume or surface of various solids. Two samples of 3D diagrams devoted to volume can be found in Fig. 8 and 9.

7 Concluding remarks

The isoperimetric problem is not a part of the secondary school curriculum, but the idea of interdependence of area and perimeter of essential plane figures should be an integral part of the content knowledge from lower-secondary school, expanding further to interdependence of volume and surface of essential solids at upper-secondary school. There are ways to engage the issue in secondary math instruction, and GeoGebra can provide an ICT support to this instruction.

Part of the issue was presented at the CADGME 2012 conference, see [6] for talk slides.

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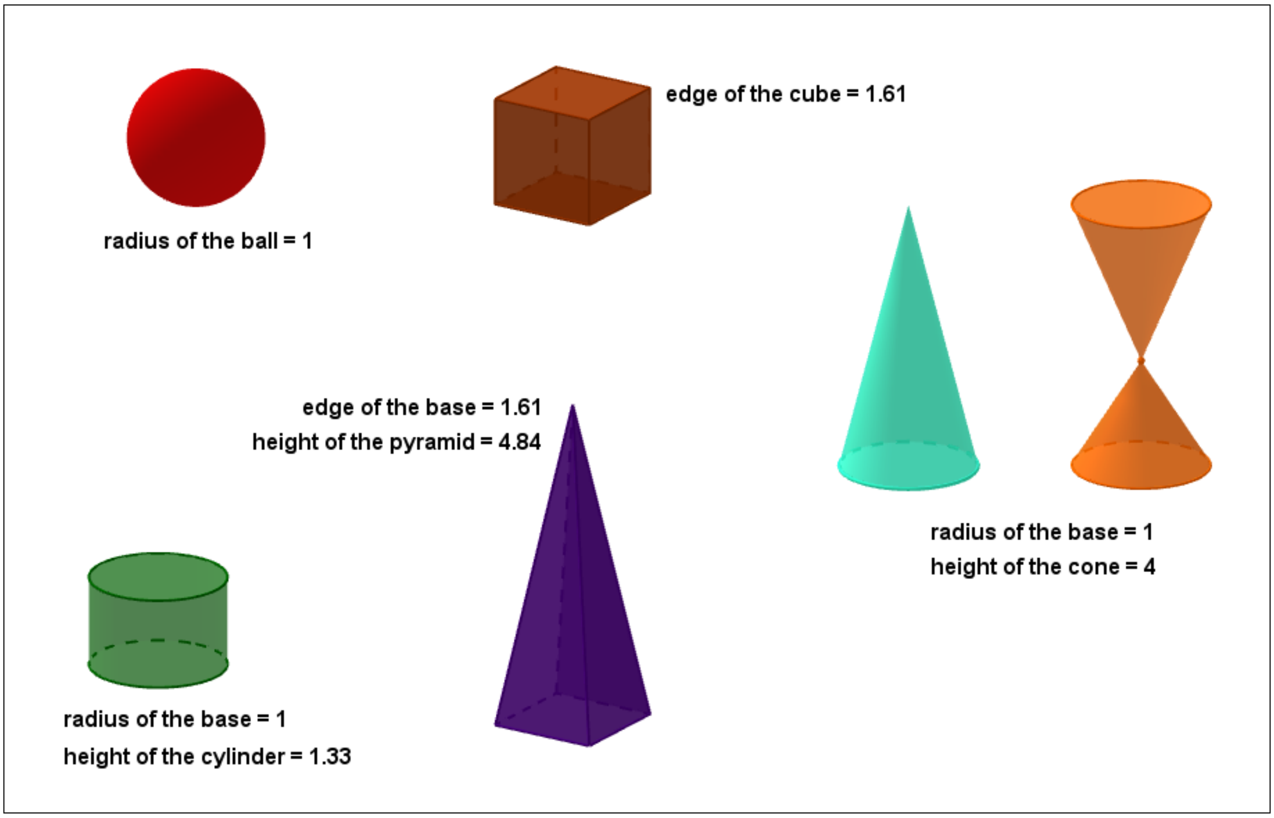

Figure 8: Various essential solids with the same volume.

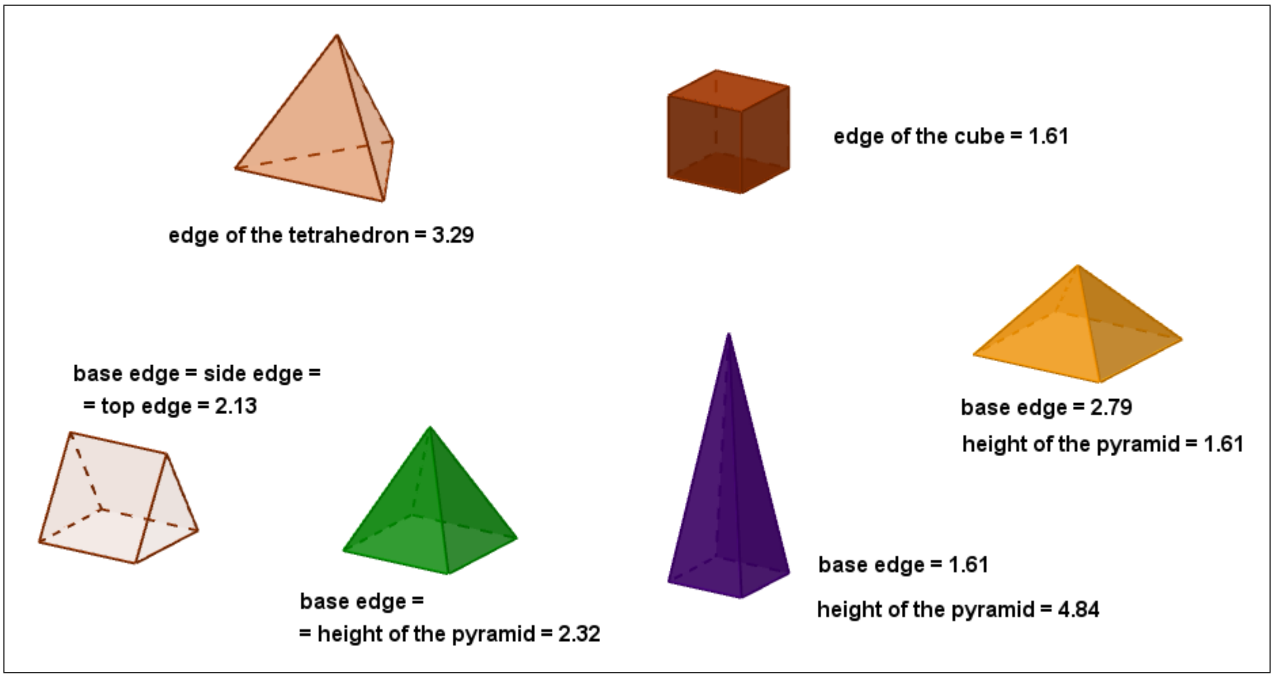

Figure 9: Various prisms with the same volume.

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